Gel'fand-Yaglom Wave Equations based on the Representation $(3/2, 5/2) \oplus (1/2, 5/2) \oplus (-1/2, 5/2) \oplus (-3/2, 5/2)$. Charge and Causality

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Z. Naturforsch. **40 a**, 1062–1065 (1985); received February 18, 1985

There are no examples of higher spin wave-equations of the Gel'fand-Yaglom form based on the representation $(3/2,\ 5/2) \oplus (1/2,\ 5/2) \oplus (-1/2,\ 5/2) \oplus (-3/2,\ 5/2)$ having definite charge and propagating causally in an external electromagnetic field.

1. Introduction

Gel'fand-Yaglom wave-equations [1, 2] are described by the system of first order differential equations

$$\mathbb{L}_0 \frac{\partial \psi}{\partial x_0} + \mathbb{L}_1 \frac{\partial \psi}{\partial x_1} + \mathbb{L}_2 \frac{\partial \psi}{\partial x_2} + \mathbb{L}_3 \frac{\partial \psi}{\partial x_3} + i \chi \psi = 0\,, \qquad (1)$$

where $\mathbb{L}_k$, $k = 0, 1, 2, 3$ are four matrices whose dimension depends on the representation chosen to describe the invariance of the equation. ψ is the wave function assuming values in the same space in which the matrices $\mathbb{L}_k$ act. Our notation is the same as that of Gel'fand, Minlos and Shapiro [3].

In this paper we shall be concerned with equations of the Gel'fand-Yaglom form for which the wave function ψ transforms according to the 20-dimensionl representation with components $\tau_1 \sim (3/2,\ 5/2)$, $\dot{\tau}_1 \sim (-3/2,\ 5/2)$, $\tau_2 \sim (1/2,\ 5/2)$, $\dot{\tau}_2 \sim (-1/2,\ 5/2)$ interlocking according to the scheme

$$\tau_1 \underset{c^{\tau_2 \tau_1}}{\overset{c^{\tau_1 \tau_2}}{\rightleftarrows}} \tau_2 \underset{c^{\dot{\tau}_2 \tau_2}}{\overset{c^{\tau_2 \dot{\tau}_2}}{\rightleftarrows}} \dot{\tau}_2 \underset{c^{\dot{\tau}_1 \dot{\tau}_2}}{\overset{c^{\dot{\tau}_2 \dot{\tau}_1}}{\rightleftarrows}} \dot{\tau}_1\,, \qquad (2)$$

where $c^{\tau_i \tau_j}$ are constants linking the different components of the representation. The maximum spin described by a wave-equation of the Gel'fand-Yaglom form based on the above representation can be 3/2. Our primary concern is to find examples of wave-equations based on the 20-dimensional representation (2) having definite charge and describing particles of spin greater than 1/2. We shall restrict ourselves to wave equations with nondiagonalizable matrix $\mathbb{L}_0$, because it can be shown that all Gel'fand-Yaglom wave-equations with diagonalizable matrix $\mathbb{L}_0$ have indefinite charge except the Dirac equation.

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2. General form of $\mathbb{L}_0$

To start with we need to find the matrix $\mathbb{L}_0$. We use as basis for expressing $\mathbb{L}_0$ the canonical basis

$$\{\xi_{lm}\} = \{\xi^{\tau_1}_{3/2,\,3/2}, \xi^{\tau_1}_{3/2,\,1/2}, \xi^{\tau_1}_{3/2,\,-1/2}, \xi^{\tau_1}_{3/2,\,-3/2}, \xi^{\tau_2}_{1/2,\,1/2}, \xi^{\tau_2}_{1/2,\,-1/2}, \xi^{\tau_2}_{3/2,\,3/2}, \xi^{\tau_2}_{3/2,\,1/2}, \xi^{\tau_2}_{3/2,\,-1/2}, \xi^{\tau_2}_{3/2,\,-3/2}, \xi^{\dot{\tau}_2}_{1/2,\,1/2}, \xi^{\dot{\tau}_2}_{1/2,\,-1/2}, \xi^{\dot{\tau}_2}_{3/2,\,3/2}, \xi^{\dot{\tau}_2}_{3/2,\,1/2}, \xi^{\dot{\tau}_2}_{3/2,\,-1/2}, \xi^{t_2}_{3/2,\,-3/2}, \xi^{\dot{\tau}_1}_{3/2,\,-3/2}, \xi^{\dot{\tau}_1}_{3/2,\,1/2}, \xi^{\dot{\tau}_1}_{3/2,\,-1/2}, \xi^{\dot{\tau}_1}_{3/2,\,-3/2}\}\,. \qquad (3)$$

The elements of $\mathbb{L}_0$ with respect to the canonical basis have the form

$$c^{\tau\tau'}_{lm,\,l'm'} = c^{\tau\tau'}_{l}\,\delta_{ll'}\,\delta_{mm'} \qquad (4)$$

(cf. Ref. [3], pp. 274–280) where the numbers $c^{\tau\tau'}_l$ are different from zero only in the case in which the components $\tau \sim (l_0, l_1)$ and $\tau' \sim (l'_0, l'_1)$ of the representation under which ψ transforms are interlocking. The numbers $c^{\tau\tau'}_l$ are given by the following formulae:

for $(l'_0, l'_1) = (l_0 + 1, l_1)$,

$$c^{\tau\tau'}_l = c^{\tau\tau'} \sqrt{(l + l_0 + 1)(l - l_0)}\,, \qquad (5)$$

$$c^{\tau'\tau}_l = c^{\tau'\tau} \sqrt{(l + l_0 + 1)(l - l_0)}\,, \qquad (6)$$

for $(l'_0, l'_1) = (l_0, l_1 + 1)$,

$$c^{\tau\tau'}_l = c^{\tau\tau'} \sqrt{(l + l_1 + 1)(l - l_1)}\,, \qquad (7)$$

$$c^{\tau'\tau}_l = c^{\tau'\tau} \sqrt{(l + l_1 + 1)(l - l_1)}\,, \qquad (8)$$

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where $c^{\tau\tau'}$ and $c^{\tau'\tau}$ are arbitrary complex numbers. In all the other cases $c^{\tau\tau'} = c^{\tau'\tau} = 0$. Similar formulae exist for the matrices $\mathbb{L}_1$, $\mathbb{L}_2$, $\mathbb{L}_3$. Using (5)–(8) and the canonical basis (3) we find that the matrix $\mathbb{L}_0$ with respect to this basis has the following blocks: for $l = 1/2$

$$\mathbb{L}_0(l=1/2) = \begin{array}{c|cc} & \tau_2 & \dot\tau_2 \\ \hline \tau_2 & 0 & c^{\tau_2\dot\tau_2} \\ \dot\tau_2 & c^{\dot\tau_2\tau_2} & 0 \end{array}, \tag{9}$$

for $l = 3/2$

$$\mathbb{L}_0(l=3/2) = \begin{array}{c|cccc} & \tau_1 & \tau_2 & \dot\tau_2 & \dot\tau_1 \\ \hline \tau_1 & 0 & \sqrt{3}\,c^{\tau_1\tau_2} & 0 & 0 \\ \tau_2 & \sqrt{3}\,c^{\tau_2\tau_1} & 0 & 2\,c^{\tau_2\dot\tau_2} & 0 \\ \dot\tau_2 & 0 & 2\,c^{\dot\tau_2\tau_2} & 0 & \sqrt{3}\,c^{\dot\tau_2\dot\tau_1} \\ \dot\tau_1 & 0 & 0 & \sqrt{3}\,c^{\dot\tau_1\dot\tau_2} & 0 \end{array}. \tag{9}$$

If invariance of the wave-equation with respect to the complete group is required the following conditions must be satisfied:

$$c^{\tau_1\tau_2} = c^{\dot\tau_1\dot\tau_2}, \quad c^{\tau_2\tau_1} = c^{\dot\tau_2\dot\tau_1}, \quad c^{\dot\tau_2\tau_2} = c^{\tau_2\dot\tau_2}. \tag{10}$$

Furthermore let us require that the above wave-equations be derivable from an invariant Lagrangian by variation. For this purpose we construct bilinear forms consistent with the representation (2). The condition for the existence of a bilinear form is that each component τ appears together with its conjugate $\dot\tau$. This condition is satisfied for the representation (2), and we have the following general expression for it:

$$(\psi_1, \psi_2) = \sum_{\substack{\tau = (\tau_1, \dot\tau_1, \tau_2, \dot\tau_2) \\ l = 1/2, 3/2 \\ m = 3/2, 1/2, -1/2, -3/2}} a^{\tau\dot\tau} s_l^{\tau\dot\tau} x^{\tau}_{lm} \bar y^{\dot\tau}_{lm}, \tag{11}$$

where $s_l^{\tau\dot\tau} = s_l^{\dot\tau\tau} = (-1)^{[l]}$ and $[l]$ is the integer part of l. An equivalent way of describing a bilinear form is by means of the matrix

$$\mathbb{A} = (-1)^{[l]} a^{\tau\dot\tau} \delta_{ll'} \delta_{mm'}, \tag{12}$$

which in the canonical basis can be cast into block form.

For the representation (2) we find the following nondegenerate (i.e. $\det \mathbb{A} \neq 0$) bilinear forms (ψ_1, ψ_2), which we give in terms of the constants $a^{\dot\tau\tau}$ defining them:

$$a^{\tau_1\dot\tau_1} = a^{\dot\tau_1\tau_1} = 1, \quad a^{\tau_2\dot\tau_2} = a^{\dot\tau_2\tau_2} = 1, \tag{13}$$

$$a^{\tau_1\dot\tau_1} = a^{\dot\tau_1\tau_1} = 1, \quad a^{\tau_2\dot\tau_2} = a^{\dot\tau_2\tau_2} = -1, \tag{14}$$

$$a^{\tau_1\dot\tau_1} = a^{\dot\tau_1\tau_1} = -1, \quad a^{\tau_2\dot\tau_2} = a^{\dot\tau_2\tau_2} = 1, \tag{15}$$

$$a^{\tau_1\dot\tau_1} = a^{\dot\tau_1\tau_1} = -1, \quad a^{\tau_2\dot\tau_2} = a^{\dot\tau_2\tau_2} = -1. \tag{16}$$

In order that a Gel'fand-Yaglom wave-equation be derivable from an invariant Lagrangian it is necessary and sufficient that the relations

$$c^{\tau\tau'} a^{\tau'\tau'^*} = \bar c^{\tau'^*\tau^*} a^{\tau\tau^*} \tag{17}$$

be satisfied among the elements of the matrices $\mathbb{L}_0$ and $\mathbb{A}$ or equivalently the relations

$$\mathbb{A}\,\mathbb{L}_0 = \mathbb{L}_0^{+}\mathbb{A} \quad \text{or} \quad \mathbb{A}\,\mathbb{L}_0^{+} = \mathbb{L}_0\mathbb{A}$$
$$(\text{since } \mathbb{A}^2 = 1 \text{ and } \mathbb{A} = \mathbb{A}^{+}). \tag{18}$$

Using these relations we find for the bilinear forms (13)–(16) the following relations satisfied among the elements of the matrix $\mathbb{L}_0$ in order that the Gel'fand-Yaglom wave-equation be derivable from an invariant Lagrangian:

First case (Bilinear form (13))

$$c^{\tau_1\tau_2} = \bar c^{\dot\tau_2\dot\tau_1}, \quad c^{\dot\tau_1\dot\tau_2} = \bar c^{\tau_2\tau_1}, \quad c^{\tau_2\dot\tau_2} = \bar c^{\tau_2\dot\tau_2} \text{ (real)}. \tag{19}$$

Second case (Bilinear form (14))

$$c^{\tau_1\tau_2} = -\bar c^{\dot\tau_2\dot\tau_1}, \quad c^{\tau_1\dot\tau_2} = -\bar c^{\tau_2\tau_1},$$
$$c^{\tau_2\dot\tau_2} = \bar c^{\tau_2\dot\tau_2} \quad \text{(real)}. \tag{20}$$

Third case (Bilinear form (15))

$$c^{\tau_1\tau_2} = -\bar c^{\dot\tau_2\dot\tau_1}, \quad c^{\dot\tau_1\dot\tau_2} = -\bar c^{\tau_2\tau_1},$$
$$c^{\tau_2\dot\tau_2} = \bar c^{\tau_2\dot\tau_2} \text{ (real)}. \tag{21}$$

Fourth case (Bilinear form (16))

$$c^{\tau_1\tau_2} = \bar c^{\dot\tau_2\dot\tau_1}, \quad c^{\dot\tau_1\dot\tau_2} = \bar c^{\tau_2\tau_1}, \quad c^{\tau_2\dot\tau_2} = \bar c^{\tau_2\dot\tau_2} \text{ (real)}. \tag{22}$$

Combining these relations with the relations of invariance of the equation under the complete group we find for the matrix $\mathbb{L}_0$ which is invariant under the complete group and derivable from an invariant Lagrangian the blocks:

First and fourth cases:

$$\mathbb{L}_0(l=1/2) = \begin{array}{c|cc} & \tau_2 & \dot\tau_2 \\ \hline \tau_2 & 0 & c^{\tau_2\dot\tau_2} \\ \dot\tau_2 & c^{\tau_2\dot\tau_2} & 0 \end{array},$$

$$\mathbb{L}_0(l=3/2) = \begin{array}{c|cccc} & \tau_1 & \tau_2 & \dot\tau_2 & \dot\tau_1 \\ \hline \tau_1 & 0 & \sqrt{3}\,c^{\tau_1\tau_2} & 0 & 0 \\ \tau_2 & \sqrt{3}\,\bar c^{\tau_1\tau_2} & 0 & 2\,c^{\tau_2\dot\tau_2} & 0 \\ \dot\tau_2 & 0 & 2\,c^{\tau_2\dot\tau_2} & 0 & \sqrt{3}\,\bar c^{\tau_1\tau_2} \\ \dot\tau_1 & 0 & 0 & \sqrt{3}\,c^{\tau_1\tau_2} & 0 \end{array}. \tag{23}$$

Second and third cases:

$$\mathbb{L}_0\,(l=1/2) = \begin{array}{c} \\ \tau_2 \\ \dot{\tau}_2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} \tau_2 & \dot{\tau}_2 \end{array} \\ \begin{pmatrix} 0 & c^{\tau_2\tau_2} \\ c^{\tau_2\dot{\tau}_2} & 0 \end{pmatrix} \end{array}, \tag{24}$$

$$\mathbb{L}_0\,(l=3/2) = \begin{array}{c|cccc} & \tau_1 & \tau_2 & \dot{\tau}_2 & \dot{\tau}_1 \\ \hline \tau_1 & 0 & \sqrt{3}\,c^{\tau_1\tau_2} & 0 & 0 \\ \tau_2 & -\sqrt{3}\,\bar{c}^{\tau_1\tau_2} & 0 & 2\,c^{\tau_2\dot{\tau}_2} & 0 \\ \dot{\tau}_2 & 0 & 2\,c^{\tau_2\dot{\tau}_2} & 0 & -\sqrt{3}\,\bar{c}^{\tau_1\tau_2} \\ \dot{\tau}_1 & 0 & 0 & \sqrt{3}\,c^{\tau_1\tau_2} & 0 \end{array}.$$

3. Charge

We study next the charge associated with the above Gel'fand-Yaglom wave equations. We recall that to each wave-equation derivable from an invariant Lagrangian function corresponds a vector with components

$$s_k = (\mathbb{L}_k\,\psi, \psi) \tag{25}$$

known as the four-current vector. Its time component

$$s_0 = (\mathbb{L}_0\,\psi, \psi) = \psi^+ \mathbb{A}\,\mathbb{L}_0\,\psi \tag{26}$$

is known as the charge density, where ψ^+ is the complex conjugate transpose of ψ. (The charge is the integral of s_0 over all space.) Gel'fand, Minlos and Shapiro give the following definition for a wave-equation to have positive charge density: The condition for a Gel'fand-Yaglom wave-equation to have positive charge density is that for all the eigenvectors ψ_λ of the matrix $\mathbb{L}_0$ with nonzero eigenvalue λ the corresponding charge density ϱ_λ is positive, i.e.

$$\varrho_\lambda = (\mathbb{L}_0\,\psi_\lambda, \psi_\lambda) > 0\,. \tag{27}$$

Likewise one can define negative charge density to mean that s_0 is either positive or negative.

As it was said earlier in this work we are looking for examples of wave-equations describing particles of spin greater than one half with definite charge. These examples are characterized by a nondiagonalizable matrix $\mathbb{L}_0$. All examples of wave-equations with matrix $\mathbb{L}_0$ of the form (23) are characterized by a diagonalizable matrix since $\mathbb{L}_0$ in this case is hermitian and so they give indefinite charge. Thus the only hope we have to find examples of wave equations with definite charge lies with case (24). We have studied thoroughly this case and our results are the following:

There are no examples of wave-equations describing particles of spin 3/2 based on the representation (2) and having definite charge according to the definition of Gel'fand, Minlos and Shapiro given above.

If we extend this definition of definite charge to include those cases for which certain charge densities vanish without the eigenvalues being zero, then we find the following two examples based on the representation (2) and having constants entering $\mathbb{L}_0$ connected as follows:

Example 1: $c^{\tau_2\dot{\tau}_2} = \sqrt{3\,c^{\tau_1\tau_2}\bar{c}^{\tau_1\tau_2}}$, $c^{\tau_1\tau_2}$ is any complex number,

Example 2: $c^{\tau_2\dot{\tau}_2} = -\sqrt{3\,c^{\tau_1\tau_2}\bar{c}^{\tau_1\tau_2}}$, $c^{\tau_1\tau_2}$ is any complex number.

The matrix $\mathbb{L}_0$ in these examples is not diagonalizable with a minimal polynomial of the form

$$m\,(\mathbb{L}_0) = \{\mathbb{L}_0 - \sqrt{3\,c^{\tau_1\tau_2}\bar{c}^{\tau_1\tau_2}}\}^2 \cdot \{\mathbb{L}_0 + \sqrt{3\,c^{\tau_1\tau_2}\bar{c}^{\tau_1\tau_2}}\}^2\,. \tag{28}$$

The charge densities in the state $l = 3/2$ vanish without the eigenvalues being zero while the charge densities in the state $l = 1/2$ are definite. These examples do not satisfy the definition of Gel'fand, Minlos and Shapiro but they can lead to an overall charge which is definite in the state $l = 1/2$.

4. A Search for Causal Wave-equations

Finally we looked for examples of wave-equations based on the representation (2) and whose matrix $\mathbb{L}_0$ satisfies the criterion of Amar and Dozzio [4, 5], i.e. the minimal polynomial of $\mathbb{L}_0$ is of the form

$$m\,(\mathbb{L}_0) = \mathbb{L}_0 \prod_{i=1}^{k} \{(\mathbb{L}_0)^2 - \lambda_i^2\}^{r_i} = 0 \tag{29}$$

(where $r_i \geqq 1$ and $\lambda_i \neq 0$ are the eigenvalues of $\mathbb{L}_0$). These equations according to Amar and Dozzio propagate causally in the presence of an external electromagnetic field.

Our results are that there are examples of wave equations which satisfy the criterion of Amar and Dozzio but they have either indefinite charge (i.e. $\mathbb{L}_0$ is diagonalizable) or the charge cannot be defined because $\mathbb{L}_0$ is not derivable from an invariant Lagrangian. (Notice that if the charge associated with a wave-equation is indefinite the electromagnetic field cannot be introduced into it by minimal coupling $p_\mu \to p_\mu + e A_\mu$, $\mu = 0, 1, 2, 3$).

5. Summary

Our general conclusion is that there are no examples of wave-equations for spin greater than 1/2 based on the 20 dimensional representation (2) having definite charge and propagating causally in an external electromagnetic field.

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